About Rolle's Theorem

Rolle's theorem is named after Michel Rolle (1652-1719), a French mathematician. Rolle discovered that theorem in 1691.

Rolle's theorem says:

If a function *f* is *continuous* on a *closed and bounded* interval $[a;b]$ and *differentiable* in the *open* interval $(a;b)$ and if, in addition, the value *f*(*a*) *is the same as the value of* *f*(*b*), then *there exists* (at least) *some point* *c* within the interval $(a;b)$ at which $f'(c)=0$, *i.e*. at that point *c* the first derivative of the funcion is going to be 0.

Notice that the function must be differentiable in the open interval $(a;b)$, that is at any point *x* so that $a<x<b$; whereas, at the endpoints *a* and *b* the funcion may or may not be differentiable.

In the statement of Rolle's theorem, as you can see, there are three conditions :

1) f must be continuos on $[a;b]$; 2) f must be differentiable in $(a;b)$; 3) *f*(*a*) = *f*(*b*)

The theorem applies if and only if all of these three conditions are met.

When using Rolle's theorem, after verifying that all of the three conditions are met, to find the value of *c* that Rolle's theorem states to exist, we first have to take the derivative of the function and then we need to set that derivative equal to 0 : $f'(x)=0$ and solve this equation.

To better understand what it is just said, let's try solving the following examples.

1. *Verify that the function* $f(x)=x^2-3x+5$ *satisfies the conditions of Rolle's theorem on the interval* $[0;3]$. *Then find all numbers c that satisfy the conclusion of Rolle's theorem itself.*

Solution.

We have a polynomial function and polynomial functions are continuous and differentiable everywhere on the set of real numbers ; so, the first two conditions are met on the given interval $[0;3]$.The next thing we need to find out is *f*(*a*) = *f*(*b*). that is *f*(*0*) = *f*(*3*). Are these two equal to each other? Let's see if they are.

So, *f*(0) is going to be

$$f(0)=0^2-3\cdot 0+5$$

which is 5 ; now, what about *f*(3) ? If we replace *x* with *3*, we get

$$f(3)=3^2-3\cdot 3+5$$

which also makes 5. Therefore also the third condition is met and what that means is that there is some value *c* where $f'(c)=0$. At this point, to find the *c* value the first step is to take the first derivative of the function. It's very easy to calculate the derivative of the given function. If you do it, you get

$$f'(x)=2x-3.$$

And now, if you replace x with c you get

$$f'(c) = 2c - 3.$$

Now we want to find the c value where the first derivative would be 0 ; so, let's set it equal to 0 and solve for c :

$$2c - 3 = 0.$$

To solve this equation we first add 3 to both sides so that on the left side -3 and +3 cancel and the equation becomes

$$2c = 3;$$

now if we divide both sides by 2, the c value we are looking for is going to be $c = \frac{3}{2}$.

Lastly, we have to make sure that this number is within the given interval.

Does the number 3/2 belong to the interval $(0;3)$? Of course it does, because it is a positive number (i.e. greater than zero) which also is less than 3. In symbols : 0 < 3/2 < 3.

We are done !

2. *Determine if Rolle's theorem applies to the function*

$$f(x) = x\sqrt{4-x}$$

on the interval $[0;4]$ *; if so, find all number c for which the theorem holds.*

Solution.

The domain of the given function is the set $D = (-\infty;\ 4]$ (closed on the right) because the quantity inside the radical must be *greater than or equal to* zero : $4 - x \geq 0$, which means $x \leq 4$. So the domain is the set of all the real numbers from negative infinity to 4 (included). Our function is then continuos on its domain and then it is continuous on the given interval $[0;4]$; so we are fine with the first condition of Rolle's theorem. Is the function differentiable in the open interval $(0;4)$? Before we answer we need to know the derivative. Let's find the derivative of the given function using the *product rule* :

$$f'(x) = 1 \cdot \sqrt{4-x} + x \cdot \frac{-1}{2\sqrt{4-x}}$$

Now we want to combine the two terms (on the right side of the equal sign) into a single term by getting the common denominator, which is $2\sqrt{4-x}$; since $\sqrt{4-x}$ times $\sqrt{4-x}$ is just $(4-x)$, we have :

$$f'(x) = \frac{2(4-x) - x}{2\sqrt{4-x}} = \frac{-3x + 8}{2\sqrt{4-x}}$$

As you can see, $f'(x)$ is undefined at $x = 4$, because the denominator cannot equal zero, then *x* must be different from 4. Does that mean Rolle's theorem doesn't apply ? Actually, Rolle's theorem requires that the function must be differentiable at every point inside the interval $(0; 4)$ which doesn't include 4, so Rolle's theorem still applies here. Now we want to test the third condition : are *f*(0) and *f*(4) equal to each other ? Let'see if they are. What does the value *f*(0) equal ? If we plug in 0 for *x* we get f(0) = 0 (0 times anything is 0). And we also get f(4) = 0 . So, Rolle's theorem applies, then we want to find the *c* value that the theorem says exists. So we need to set the derivative equal to zero. As we know, when the numerator is zero the entire fraction is zero, so we just need to require that

$$-3x + 8 = 0.$$

If we add -8 to both sides we get $-3x = -8$; now if we divide both sides by -3 we get $x = \frac{8}{3}$, which is about 2.67 and then it is in the interval $(0; 4)$. So, the *c* value for which Rolle's theorem holds is $c = \frac{8}{3}$.

3. *Let f be a function defined by*

$$f(x) = \frac{2x - 1}{x - 3}$$

Answer the following question : *does Rolle's theorem apply to that function on the interval* $[1; 5]$?

Answer.

The given function is defined for all real numbers except at $x = 3$ because if we put in 3 for *x* the denominator would be 0. Then our function is not continuos at $x = 3$ ($x = 3$ is a point of discontinuity and the function has a vertical asymptote since the limit of $f(x)$ as *x* tends to 3 is ∞).

As we can see, the value $x = 3$ belongs to the given interval $[1; 5]$ and then Rolle's theorem does not apply.