ROLLE'S THEOREM

Statement and proof

Rolle's Theorem is about real valued functions which are continuous and differentiable on an interval. Here is a statement of the theorem:

Let f be a function defined on a closed and bounded interval $[a;b]$; if f is continuous on $[a;b]$ and differentiable in the open interval $(a;b)$ and, in addition, $f(a)=f(b)$ (i.e. the function's values at the two endpoints *a* and *b* are the same), *then there is* (at least) *some point c in $(a;b)$ at which $f'(c)=0$.*

Proof.

Since f is continuous on the closed interval $[a;b]$, by the Extreme Value Theorem (also known as Weierstrass' theorem) we know that f has a maximum value *M* and a minimum value *m* on the closed interval $[a;b]$, that is there exist two points x_M , $x_m \in [a;b]$ such that $f(x_M) = M$ and $f(x_m) = m$ with

$$f(x_m) \leq f(x) \leq f(x_M)$$

for all $x \in [a;b]$.

There are two possibilities: either $M = m$ or $M \neq m$.

First we suppose the maximum value *M* equals the minimum value *m*. As a consequence, all values of f on $[a;b]$ are equal, and then f is constant on $[a;b]$. Since the derivative of any constant function is always zero, then $f'(x) = 0$ for all *x* in $(a;b)$, so one may take *c* to be anything in $(a;b)$. Actually, this a trivial case: Rolle's theorem holds for all *x* values in $(a;b)$.

Now we suppose that *M* is not equal to *m*. So, at least one of *M* and *m* is not equal to the value $f(a) = f(b)$.

We first consider the case where the maximum value $M \neq f(a) = f(b)$, so the point x_M lies inside the interval $(a;b)$. Since at x_M the function has a maximum and it is there differentiable (by hypothesis), Fermat's theorem says that the first derivative at $x = x_M$ is going to be 0.

So, we take $c = x_M$, and get $f'(c) = f'(x_M) = 0$ and we are done with this case.

The case with the minimum value $m \neq f(a) = f(b)$ is similar and left for you to do. And this completes the proof of Rolle's Theorem.

Notice that Rolle's theorem guarantees that there exists at least one *c* value inside the interval $(a;b)$ at which the first derivative would be zero; it doesn't say that such a value is unique: there could be more than just one point where the derivative would be zero. Rolle's theorem does not tell us how many they are or how to find them.

Geometric interpretation of Rolle's theorem.

Geometrically, as we know, the first derivative $f'(c)$ gives us the slope of the tangent line to the graph of the function f at the point $(c; f(c))$. So, what Rolle's theorem says is that if all hypotheses are satisfied, then at some point $(c; f(c))$ the tangent line to the graph is going to be parallel to the *x*-axis because its slope

$$m = f'(c)$$

would be zero.